

**GLOBAL IN TIME ESTIMATES FOR ONE-DIMENSIONAL
COMPRESSIBLE VISCOUS MICROPOLAR FLUID MODEL**

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ABSTRACT. An initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodynamically perfect and polytropic. A problem has a unique strong solution on $]0, 1[\times]0, T[$, for each $T > 0$. Using this result we obtain a priori estimates for the solution independent of T .

1. INTRODUCTION

Micropolar fluids are fluids with microstructure. They belong to a class of fluids with nonsymmetric stress tensor that we call polar fluids. The model of micropolar fluids is introduced in [3] by C.A. Eringen as a significant generalization of the classical Navier-Stokes model. Some problems for three-dimensional flow of an incompressible viscous micropolar fluid are presented in the book [6], in particular their mathematical theories.

We consider nonstationary one-dimensional flow of a compressible and heat-conducting micropolar fluid. The equations of motion for this fluid are derived from integral form of conservation laws for polar fluids, under a number of supplementary assumptions such as selection of constitutive equations, Fourier's law, Boyle's law and polytropy (see [8]). A corresponding initial-boundary value problem has a unique strong solution on $]0, 1[\times]0, T[$, for each $T > 0$ ([9]). Considering stabilisation problem, one has to prove a priori estimates for the solution independent of T , what is the main difficulty. We obtain these estimates using the results from [9]. In our proof we follow some ideas of S.N. Antontsev, A.V. Kazhykhov and V.N. Monakhov, applied to 1-D initial-boundary value problem for a classical fluid ([1]).

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2. STATEMENT OF THE PROBLEM AND THE MAIN RESULT

Let ρ, v, ω and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangean description.

Governing equations of the flow under consideration are as follows ([9]):

$$(2.1) \quad \frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0,$$

$$(2.2) \quad \frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left(\rho \frac{\partial v}{\partial x} \right) - K \frac{\partial}{\partial x} (\rho \theta),$$

$$(2.3) \quad \rho \frac{\partial \omega}{\partial t} = A \left[\rho \frac{\partial}{\partial x} \left(\rho \frac{\partial \omega}{\partial x} \right) - \omega \right],$$

$$(2.4) \quad \rho \frac{\partial \theta}{\partial t} = -K \rho^2 \theta \frac{\partial v}{\partial x} + \rho^2 \left(\frac{\partial v}{\partial x} \right)^2 + \rho^2 \left(\frac{\partial \omega}{\partial x} \right)^2 + \omega^2 + D \rho \frac{\partial}{\partial x} \left(\rho \frac{\partial \theta}{\partial x} \right)$$

in $]0, 1[\times \mathbb{R}^+$, where K, A and D are positive constants. We take the homogeneous boundary conditions

$$(2.5) \quad v(0, t) = v(1, t) = 0,$$

$$(2.6) \quad \omega(0, t) = \omega(1, t) = 0,$$

$$(2.7) \quad \frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(1, t) = 0$$

for $t > 0$ and non-homogeneous initial conditions

$$(2.8) \quad \rho(x, 0) = \rho_0(x),$$

$$(2.9) \quad v(x, 0) = v_0(x),$$

$$(2.10) \quad \omega(x, 0) = \omega_0(x),$$

$$(2.11) \quad \theta(x, 0) = \theta_0(x)$$

for $x \in \Omega =]0, 1[$, where ρ_0, v_0, ω_0 and θ_0 are given functions. We assume that there exist $m, M \in \mathbb{R}^+$, such that

$$(2.12) \quad 0 < m \leq \rho_0(x) \leq M, \quad m \leq \theta_0(x) \leq M, \quad x \in \Omega.$$

Let

$$(2.13) \quad \rho_0, \theta_0 \in H^1(\Omega), \quad v_0, \omega_0 \in H_0^1(\Omega).$$

Then for each $T > 0$ the problem (2.1)-(2.11) has a unique strong solution ([9])

$$(2.14) \quad (x, t) \rightarrow (\rho, v, \omega, \theta)(x, t), \quad (x, t) \in Q_T = \Omega \times]0, T[$$

with the properties:

$$(2.15) \quad \rho \in L^\infty(0, T; H^1(\Omega)) \cap H^1(Q_T),$$

$$(2.16) \quad v, \omega, \theta \in L^\infty(0, T; H^1(\Omega)) \cap H^1(Q_T) \cap L^2(0, T; H^2(\Omega)),$$

$$(2.17) \quad \rho > 0, \theta > 0 \text{ on } \bar{Q}_T.$$

From embedding and interpolation theorems ([5]) one can conclude that from (2.15) and (2.16) it follows:

$$(2.18) \quad \rho \in L^\infty(0, T; C(\bar{\Omega})) \cap C([0, T]; L^2(\Omega)),$$

$$(2.19) \quad v, \omega, \theta \in L^2(0, T; C^1(\bar{\Omega})) \cap C([0, T], H^1(\Omega)),$$

$$(2.20) \quad \rho, v, \omega, \theta \in C(\bar{Q}_T).$$

Our purpose is to prove the following result.

THEOREM 2.1. *Let the initial functions satisfy (2.12) and (2.13). Then the problem (2.1)-(2.11) has a solution*

$$(2.21) \quad (x, t) \rightarrow (\rho, v, \omega, \theta)(x, t), \quad (x, t) \in Q = \Omega \times]0, \infty[,$$

having the properties:

$$(2.22) \quad \rho \in L^\infty(0, \infty; H^1(\Omega)),$$

$$(2.23) \quad \frac{\partial \rho}{\partial t} \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(Q), \quad \frac{\partial \rho}{\partial x} \in L^2(0, \infty; L^2(\Omega))$$

$$(2.24) \quad v, \omega \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)) \cap H^1(Q),$$

$$(2.25) \quad \theta \in L^\infty(0, \infty; H^1(\Omega)),$$

$$(2.26) \quad \frac{\partial \theta}{\partial x} \in L^2(0, \infty; H^1(\Omega)), \quad \frac{\partial \theta}{\partial t} \in L^2(Q).$$

3. SOME PROPERTIES OF THE SOLUTION (2.14)

In that what follows we denote by $C > 0$ a generic constant, not depending of $T > 0$ and having possibly different values at different places. We use some of considerations from [1, 9, 10]. In these cases we omit proofs, making reference to correspondent pages of [1, 9, 10]. Let

$$(3.1) \quad \frac{1}{2} \|v_0\|^2 + \frac{1}{2A} \|\omega_0\|^2 + \|\theta_0\|_{L^1(\Omega)} = E_1,$$

$$(3.2) \quad \frac{1}{2A} \|\omega_0\|^2 = E_2,$$

where $||| = |||_{L^2(\Omega)}$.

LEMMA 3.1. ([9], pp. 201, 205) For $t > 0$ it holds

$$(3.3) \quad \frac{1}{2} \|v(., t)\|^2 + \frac{1}{2A} \|\omega(., t)\|^2 + \|\theta(., t)\|_{L^1(\Omega)} = E_1,$$

$$(3.4) \quad \frac{1}{2A} \|\omega(., t)\|^2 + \int_0^t \int_0^1 \rho \left(\frac{\partial \omega}{\partial x} \right)^2 dx d\tau + \int_0^t \int_0^1 \frac{\omega^2}{\rho} dx d\tau = E_2.$$

Let

$$(3.5) \quad I_0 = \int_0^1 \left[\frac{1}{2K} v_0^2 + \frac{1}{2AK} \omega_0^2 + \rho_0^{-1} (\rho_0 \ln \rho_0 - \rho_0 + 1) + \frac{1}{K} (\theta_0 - \ln \theta_0 - 1) \right] dx,$$

$$(3.6) \quad U(t) = \int_0^1 \left[\frac{1}{2K} v^2 + \frac{1}{2AK} \omega^2 + \rho^{-1} (\rho \ln \rho - \rho + 1) + \frac{1}{K} (\theta - \ln \theta - 1) \right] dx,$$

$$(3.7) \quad V(t) = \frac{1}{K} \int_0^1 \left[\frac{\rho}{\theta} \left(\frac{\partial v}{\partial x} \right)^2 + \frac{\rho}{\theta} \left(\frac{\partial \omega}{\partial x} \right)^2 + \frac{\omega^2}{\rho \theta} + D \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 \right] dx.$$

LEMMA 3.2 ([10]). For $t > 0$ it holds

$$(3.8) \quad U(t) + \int_0^t V(\tau) d\tau = I_0.$$

Let

$$(3.9) \quad \int_0^1 \frac{1}{\rho_0(x)} dx = \alpha,$$

$$(3.10) \quad \sigma = \rho \frac{\partial v}{\partial x} - K \rho \theta.$$

LEMMA 3.3 ([1], pp. 84-85). For each $t > 0$ there exists at least one point $x_0 = x_0(t) \in \bar{\Omega}$ such that

$$(3.11) \quad \int_0^t \sigma(x_0(t), \tau) d\tau = -\frac{1}{\alpha} \int_0^t \int_0^1 (v^2 + K\theta) dx d\tau \\ + \frac{1}{\alpha} \int_0^1 v_0(x) \int_x^1 \frac{d\xi}{\rho_0(\xi)} dx - \int_0^{x_0(t)} v_0(\xi) d\xi.$$

Let

$$(3.12) \quad m_\rho(t) = \min_{x \in \bar{\Omega}} \rho(x, t), \quad M_\theta(t) = \max_{x \in \bar{\Omega}} \theta(x, t)$$

and

$$(3.13) \quad G(t) = \int_0^1 \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 dx.$$

LEMMA 3.4. *There exists a constant $C \in \mathbb{R}^+$ such that for each $t > 0$ the inequality*

$$(3.14) \quad M_\theta(t) \leq C(1 + m_\rho^{-1}(t)G(t))$$

holds true.

PROOF. Let

$$(3.15) \quad \psi(x, t) = \sqrt{\theta(x, t)} - \int_0^1 \sqrt{\theta(\xi, t)} d\xi.$$

Then for $t > 0$ it holds $\int_0^1 \psi(x, t) dx = 0$ and therefore there exists $x_1 = x_1(t) \in \bar{\Omega}$ such that $\psi(x_1(t), t) = 0$. With the help of (3.15) and the Hölder inequality we find that

$$(3.16) \quad \begin{aligned} \psi(x, t) &= \int_{x_1}^x \frac{\partial}{\partial \xi} \psi(\xi, t) d\xi \leq \frac{1}{2} \int_0^1 \frac{1}{\sqrt{\theta(\xi, t)}} \left| \frac{\partial \theta}{\partial \xi} \right| d\xi \\ &\leq \frac{1}{2} \left(\int_0^1 \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial \xi} \right)^2 d\xi \right)^{1/2} \left(\int_0^1 \rho^{-1} \theta d\xi \right)^{1/2} \end{aligned}$$

and hence

$$(3.17) \quad \theta(x, t) \leq C \left(\int_0^1 \theta(\xi, t) d\xi + \int_0^1 \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial \xi} \right)^2 d\xi \cdot m_\rho^{-1}(t) \int_0^1 \theta(\xi, t) d\xi \right).$$

Taking into account (3.3) and (3.13) from (3.17) we obtain (3.14). \square

LEMMA 3.5. *The following equality holds*

$$(3.18) \quad \rho(x, t) = \frac{\rho_0(x) \exp\left\{\frac{1}{\alpha} \int_0^t \int_0^1 (v^2 + K\theta) dx d\tau + A(x, t)\right\}}{1 + K\rho_0(x) \int_0^t \theta(x, \tau) \exp\left\{\frac{1}{\alpha} \int_0^\tau \int_0^1 (v^2 + K\theta) dx ds + A(x, \tau)\right\} d\tau}$$

where

$$(3.19) \quad A(x, t) = \int_0^x v_0(\xi) d\xi - \int_{x_0(t)}^x v(\xi, t) d\xi - \frac{1}{\alpha} \int_0^1 v_0(x) \int_x^1 \frac{d\xi}{\rho_0(\xi)} dx.$$

PROOF. Taking into account (3.10) from (2.2) we get

$$(3.20) \quad \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}.$$

Integrating (3.20) over $]0, t[$, $t > 0$ and (for a given t) over $]x_0(t), x[$, $x \in [0, 1]$, (where $x_0(t)$ is defined in Lemma 3.3) we obtain

$$(3.21) \quad \int_{x_0(t)}^x (v - v_0) d\xi = \int_0^t \sigma(x, \tau) d\tau - \int_0^t \sigma(x_0(t), \tau) d\tau.$$

Using (2.1) for the function σ we have

$$(3.22) \quad \int_0^t \sigma d\tau = - \int_0^t \frac{1}{\rho} \frac{\partial \rho}{\partial t} d\tau - K \int_0^t \rho \theta d\tau = - \ln \frac{\rho}{\rho_0} - K \int_0^t \rho \theta d\tau.$$

With the help of (3.11) and (3.22) from (3.21) we get

$$(3.23) \quad \rho(x, t) \exp\left\{K \int_0^t \rho \theta d\tau\right\} = \rho_0(x) \exp\left\{\frac{1}{\alpha} \int_0^t \int_0^1 (v^2 + K\theta) dx d\tau + A(x, t)\right\}.$$

Multiplying (3.23) by $K\theta$ and integrating over $]0, t[, t > 0$, we have

$$(3.24) \quad \exp\left\{K \int_0^t \rho \theta d\tau\right\} = 1 + K \rho_0(x) \int_0^t \theta(x, \tau) \exp\left\{\frac{1}{\alpha} \int_0^\tau \int_0^1 (v^2 + K\theta) dx ds + A(x, \tau)\right\} d\tau.$$

From (3.24) and (3.23) it follows (3.18). \square

LEMMA 3.6. *There exists a constant $C \in R^+$ such that for each $t > 0$ it holds*

$$(3.25) \quad m_\rho(t) \geq C.$$

PROOF. Let $K \geq 2$. We introduce the non-negative function $B_1(t), t > 0$, by

$$(3.26) \quad B_1(t) = 2\left(\int_0^t \int_0^1 \rho \left(\frac{\partial \omega}{\partial x}\right)^2 dx d\tau + \int_0^t \int_0^1 \frac{\omega^2}{\rho} dx d\tau\right) + (K - 2) \int_0^1 \theta dx.$$

With the help of (3.4) from (3.3) we obtain

$$(3.27) \quad \|v(., t)\|^2 + K \int_0^1 \theta(x, t) dx = C + B_1(t).$$

Taking into account (3.27), (2.12), (3.12) and the estimate $|A(x, t)| \leq 2\|v_0\| + \|v(., t)\| \leq C$ from (3.18) we find the inequality

$$(3.28) \quad \rho^{-1}(x, t) \leq \frac{C(1 + \exp\{\frac{1}{\alpha} \int_0^t B_1(s) ds\} \int_0^t M_\theta(\tau) \exp\{\frac{1}{\alpha} C\tau\} d\tau)}{\exp\{\frac{1}{\alpha} Ct\} \exp\{\frac{1}{\alpha} \int_0^t B_1(\tau) d\tau\}} \leq C \exp\left\{-\frac{1}{\alpha} Ct\right\} \left(1 + \int_0^t M_\theta(\tau) \exp\left\{\frac{1}{\alpha} C\tau\right\} d\tau\right).$$

Using (3.14) from (3.28) it follows

$$(3.29) \quad m_\rho^{-1}(t) \leq C(1 + \exp\{-\frac{1}{\alpha}Ct\} \int_0^t m_\rho^{-1}(\tau)G(\tau) \exp\{\frac{1}{\alpha}C\tau\}d\tau) \\ \leq C(1 + \int_0^t m_\rho^{-1}(\tau)G(\tau)d\tau)$$

or, because of Gronwall's inequality,

$$(3.30) \quad m_\rho^{-1}(t) \leq C \exp\{\int_0^t G(\tau)d\tau\}.$$

With the help of (3.13) and (3.8) we conclude that $G \in L^1([0, \infty[)$. Then the estimate (3.25) follows immediately from (3.30).

Let now be $K < 2$ and

$$(3.31) \quad B_2(t) = K(\int_0^t \int_0^1 \rho(\frac{\partial \omega}{\partial x})^2 dx d\tau + \int_0^t \int_0^1 \frac{\omega^2}{\rho} dx d\tau) + \frac{2-K}{2} \|v(., t)\|^2.$$

Using (3.3) and (3.4) we obtain

$$(3.32) \quad \|v(., t)\|^2 + K \int_0^1 \theta(x, t) dx = C + B_2(t).$$

We have again $B_2(t) \geq 0, t > 0$. Inserting (3.32) into (3.18) in the same way as before we get (3.25). \square

Let

$$(3.33) \quad M_\rho(t) = \max_{x \in \Omega} \rho(x, t), \quad m_\theta(t) = \min_{x \in \Omega} \theta(x, t).$$

LEMMA 3.7. *There exists a constant $C \in R^+$ such that for each $t > 0$ it holds*

$$(3.34) \quad M_\rho(t) \leq C.$$

PROOF. With the help of the Hölder inequality we obtain

$$(3.35) \quad v^2(x, t) \leq (\int_0^1 |\frac{\partial v}{\partial x}| dx)^2 \leq \int_0^1 \frac{\rho}{\theta} (\frac{\partial v}{\partial x})^2 dx \cdot m_\rho^{-1}(t) \int_0^1 \theta dx,$$

$$(3.36) \quad \omega^2(x, t) \leq (\int_0^1 |\frac{\partial \omega}{\partial x}| dx)^2 \leq \int_0^1 \frac{\rho}{\theta} (\frac{\partial \omega}{\partial x})^2 dx \cdot m_\rho^{-1}(t) \int_0^1 \theta dx$$

and using (3.25), (3.3), (3.7) and (3.8) we conclude that there exists $C \in R^+$ such that for each $t > 0$ it holds

$$(3.37) \quad \int_0^t \max_{x \in \Omega} v^2(x, \tau) d\tau \leq C, \quad \int_0^t \max_{x \in \Omega} \omega^2(x, \tau) d\tau \leq C.$$

Now, we write the equality (3.3) in the form

$$(3.38) \quad \int_0^1 \theta(x, t) dx = E_1 - \frac{1}{2} \|v(\cdot, t)\|^2 - \frac{1}{2A} \|\omega(\cdot, t)\|^2$$

and with the help of (3.37) we conclude that

$$(3.39) \quad E_1 t - C \leq \int_0^t \int_0^1 \theta(x, \tau) dx d\tau \leq E_1 t + C.$$

Using the estimates (2.12), (3.37) and (3.39) from (3.18) we obtain

$$(3.40) \quad M_\rho(t) \leq \frac{C \exp\{\frac{KE_1}{\alpha} t\}}{1 + \int_0^t m_\theta(\tau) \exp\{\frac{KE_1}{\alpha} \tau\} d\tau}.$$

Now we intend to get the estimate for the function $m_\theta(t)$, $t > 0$. Therefore we introduce the function ψ_1 by

$$(3.41) \quad \psi_1(m_\theta(t), h(t)) = \int_{m_\theta(t)}^{h(t)} \frac{(s - \ln s)^{1/2}}{s} ds,$$

where $h(t) = E_1 - \frac{1}{2} \|v(\cdot, t)\|^2 - \frac{1}{2A} \|\omega(\cdot, t)\|^2 = \int_0^1 \theta(x, t) dx$. One can easily conclude that it holds

$$(3.42) \quad \psi_1(m_\theta(t), h(t)) \geq \frac{1}{E_1} (h(t) - m_\theta(t)).$$

Taking into account that there exist $x_1(t), x_2(t) \in \bar{\Omega}$ such that $m_\theta(t) = \theta(x_1(t), t)$ and $h(t) = \theta(x_2(t), t)$ and applying the Hölder inequality, from (3.41) and (3.42) we find that

$$(3.43) \quad \begin{aligned} \frac{1}{E_1} (h(t) - m_\theta(t)) &\leq \psi_1(h(t), m_\theta(t)) = \int_{m_\theta}^{h(t)} \frac{(s - \ln s)^{1/2}}{s} ds \\ &= \int_{x_1}^{x_2} \frac{(\theta(x, t) - \ln \theta(x, t))^{1/2}}{\theta(x, t)} \cdot \frac{\partial \theta}{\partial x} dx \\ &\leq \left(\int_0^1 (\theta - \ln \theta) dx \right)^{1/2} \cdot \left(\int_0^1 \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 dx \right)^{1/2} \cdot m_\rho^{-1/2}(t). \end{aligned}$$

Using (3.6), (3.8), (3.13) and (3.25) from (3.43) we get immediately

$$(3.44) \quad m_\theta(t) \geq h(t) - CG^{1/2}(t), \quad t \geq 0.$$

Applying the Young inequality with a parameter $\varepsilon > 0$ on the second term on the right-hand side of (3.44) we have

$$(3.45) \quad m_\theta(t) \geq E_1 - \frac{1}{2} \|v(\cdot, t)\|^2 - \frac{1}{2A} \|\omega(\cdot, t)\|^2 - \varepsilon - C_\varepsilon G(t).$$

Because $m_\theta(t) > 0$, $t > 0$, we conclude that

$$(3.46) \quad m_\theta(t) \geq c - \min\{c, f_0(t)\},$$

where

$$(3.47) \quad c = E_1 - \varepsilon, \quad f_0(t) = \frac{1}{2} \|v(\cdot, t)\|^2 + \frac{1}{2A} \|\omega(\cdot, t)\|^2 + C_\varepsilon G(t) \in L^1([0, \infty[).$$

Obviously,

$$(3.48) \quad f_1(t) = \min\{c, f_0(t)\} \in L^1([0, \infty[).$$

Let $\varepsilon > 0$ be such that $0 < \frac{c\alpha}{KE_1} < 1$. Using (3.48) and (3.46) from (3.40) we obtain

$$(3.49) \quad M_\rho(t) \leq \frac{C \exp\{\frac{KE_1}{\alpha}t\}}{1 + \frac{c\alpha}{KE_1}(\exp\{\frac{KE_1}{\alpha}t\} - 1) - \int_0^t f_1(\tau) \exp\{\frac{KE_1}{\alpha}\tau\} d\tau} \\ \leq \frac{C}{1 - \frac{KE_1}{c\alpha}\varphi(t)}$$

where

$$(3.50) \quad \varphi(t) = \exp\{-\frac{KE_1}{\alpha}t\} \int_0^t f_1(\tau) \exp\{\frac{KE_1}{\alpha}\tau\} d\tau.$$

One can easily conclude that $\lim_{t \rightarrow \infty} \varphi(t) = 0$. Because of it a function φ that is non-negative and continuous on $]0, \infty[$, takes its maximum at a point $t_0 \in]0, \infty[$. It holds

$$(3.51) \quad \varphi(t_0) = \sup_{t>0} \varphi(t) = \exp\{-\frac{KE_1}{\alpha}t_0\} \int_0^{t_0} f_1(\tau) \exp\{\frac{KE_1}{\alpha}\tau\} d\tau \\ \leq \exp\{-\frac{KE_1}{\alpha}t_0\} \frac{c\alpha}{KE_1} (\exp\{\frac{KE_1}{\alpha}t_0\} - 1) < \frac{c\alpha}{KE_1}$$

and we get

$$(3.52) \quad 1 - \frac{KE_1}{c\alpha}\varphi(t_0) > 0.$$

The estimate (3.34) follows from (3.52) and (3.49). \square

4. PROOF OF THEOREM 2.1.

The proof is based on a priori estimates (obtained in Section 3) for the function (2.14) not depending of T .

LEMMA 4.1. *It holds*

$$(4.1) \quad \rho \in L^\infty(Q),$$

$$(4.2) \quad v, \omega \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; L^2(\Omega)),$$

$$(4.3) \quad \frac{\partial \omega}{\partial x} \in L^2(0, \infty; L^2(\Omega)),$$

$$(4.4) \quad \theta \in L^\infty(0, \infty; L^2(\Omega)),$$

$$(4.5) \quad \frac{\partial \theta}{\partial x} \in L^2(0, \infty; L^2(\Omega)).$$

PROOF. The conclusions (4.1), (4.2) and (4.3) follow immediately from (3.3), (3.4), (3.37), (3.25) and (3.34).

As in [9] we introduce a total energy

$$(4.6) \quad \phi = \frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta.$$

We multiply the equations (2.2), (2.3) and (2.4) respectively by v , $A^{-1}\rho^{-1}\omega$ and ρ^{-1} . After addition of the obtained equations, we find that

$$(4.7) \quad \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \left(D\rho \frac{\partial \theta}{\partial x} + \rho v \frac{\partial v}{\partial x} + \rho \omega \frac{\partial \omega}{\partial x} \right) - K \frac{\partial}{\partial x} (\rho \theta v).$$

Multiplying (4.7) by ϕ and integrating over $]0, 1[$ we obtain

$$(4.8) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi(\cdot, t)\|^2 + \int_0^1 \rho \left(D \frac{\partial \theta}{\partial x} + v \frac{\partial v}{\partial x} + \omega \frac{\partial \omega}{\partial x} \right) \left(v \frac{\partial v}{\partial x} + \frac{1}{A} \omega \frac{\partial \omega}{\partial x} + \frac{\partial \theta}{\partial x} \right) dx \\ = K \int_0^1 \rho \theta v^2 \frac{\partial v}{\partial x} dx + \frac{K}{A} \int_0^1 \rho \theta v \omega \frac{\partial \omega}{\partial x} dx + K \int_0^1 \rho v \theta \frac{\partial \theta}{\partial x} dx. \end{aligned}$$

From (4.8) it follows

$$(4.9) \quad \begin{aligned} \frac{d}{dt} \|\phi(\cdot, t)\|^2 + 2D \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx + 2 \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{2}{A} \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx \\ = -2(D+1) \int_0^1 \rho \frac{\partial \theta}{\partial x} v \frac{\partial v}{\partial x} dx - 2 \left(\frac{D}{A} + 1 \right) \int_0^1 \rho \frac{\partial \theta}{\partial x} \omega \frac{\partial \omega}{\partial x} dx \\ - 2 \left(\frac{1}{A} + 1 \right) \int_0^1 \rho \omega \frac{\partial \omega}{\partial x} v \frac{\partial v}{\partial x} dx + 2K \int_0^1 \rho \theta v^2 \frac{\partial v}{\partial x} dx \\ + \frac{2K}{A} \int_0^1 \rho \theta v \omega \frac{\partial \omega}{\partial x} dx + 2K \int_0^1 \rho v \theta \frac{\partial \theta}{\partial x} dx. \end{aligned}$$

Using the Young inequality with a parameter $\varepsilon_i > 0$ ($i = 1, 2, \dots, 6$), for the terms on the right-hand side of (4.9) we find the estimates as follows:

$$(4.10) \quad \left| \int_0^1 \rho \frac{\partial \theta}{\partial x} v \frac{\partial v}{\partial x} dx \right| \leq \frac{\varepsilon_1^2}{2} \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{\varepsilon_1^{-2}}{2} \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx,$$

$$(4.11) \quad \left| \int_0^1 \rho \frac{\partial \theta}{\partial x} \omega \frac{\partial \omega}{\partial x} dx \right| \leq \frac{\varepsilon_2^2}{2} \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + \frac{\varepsilon_2^{-2}}{2} \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx,$$

$$(4.12) \quad \left| \int_0^1 \rho \omega \frac{\partial \omega}{\partial x} v \frac{\partial v}{\partial x} dx \right| \leq \frac{\varepsilon_3^2}{2} \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{\varepsilon_3^{-2}}{2} \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx,$$

$$(4.13) \quad \left| \int_0^1 \rho \theta v^2 \frac{\partial v}{\partial x} dx \right| \leq \frac{\varepsilon_4^2}{2} \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{\varepsilon_4^{-2}}{2} \int_0^1 \rho \theta^2 v^2 dx,$$

$$(4.14) \quad \left| \int_0^1 \rho \theta v \omega \frac{\partial \omega}{\partial x} dx \right| \leq \frac{\varepsilon_5^2}{2} \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + \frac{\varepsilon_5^{-2}}{2} \int_0^1 \rho \theta^2 v^2 dx,$$

$$(4.15) \quad \left| \int_0^1 \rho v \theta \frac{\partial \theta}{\partial x} dx \right| \leq \frac{\varepsilon_6^2}{2} \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx + \frac{\varepsilon_6^{-2}}{2} \int_0^1 \rho \theta^2 v^2 dx.$$

Taking $\varepsilon_6, \varepsilon_1$ and ε_2 such that

$$(4.16) \quad K \varepsilon_6^2 + (D+1) \varepsilon_1^{-2} + (DA^{-1}+1) \varepsilon_2^{-2} = D$$

and using (4.10)–(4.15), from (4.9) we find that

$$(4.17) \quad \begin{aligned} \frac{d}{dt} \|\phi(\cdot, t)\|^2 + D \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx + 2 \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{2}{A} \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx \\ \leq C_1 \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + C_2 \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + C_3 \int_0^1 \rho \theta^2 v^2 dx. \end{aligned}$$

Now, multiplying the equation (2.2) by $4v^3$, integrating over $]0, 1[$ and applying the Young inequality with a parameter $\varepsilon > 0$ we get

$$(4.18) \quad \begin{aligned} \frac{d}{dt} \int_0^1 v^4 dx + 12 \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx = 12K \int_0^1 \rho \theta v^2 \frac{\partial v}{\partial x} dx \\ \leq \frac{\varepsilon^2}{2} \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{144K^2 \varepsilon^{-2}}{2} \int_0^1 \rho \theta^2 v^2 dx. \end{aligned}$$

Setting $\varepsilon^2 = 2$ in (4.18) we have

$$(4.19) \quad \frac{d}{dt} \int_0^1 v^4 dx + 11 \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx \leq C \int_0^1 \rho \theta^2 v^2 dx.$$

Multiplying the equation (2.3) by $4\omega^3 \rho^{-1} A^{-1}$ and integrating over $]0, 1[$ we obtain easily

$$(4.20) \quad \frac{1}{A} \frac{d}{dt} \int_0^1 \omega^4 dx + 12 \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + 4 \int_0^1 \frac{\omega^4}{\rho} dx = 0.$$

Multiplying (4.19) and (4.20) respectively by $\frac{C_1}{11}$ and $\frac{C_2}{12}$, after addition of the obtained inequalities with (4.17) we find that

$$(4.21) \quad \begin{aligned} \frac{d}{dt} \|\phi(\cdot, t)\|^2 + \frac{C_1}{11} \frac{d}{dt} \int_0^1 v^4 dx + \frac{C_2}{12A} \frac{d}{dt} \int_0^1 \omega^4 dx + D \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx \\ + 2 \int_0^1 \rho v^2 \left(\frac{\partial v}{\partial x} \right)^2 dx + \frac{2}{A} \int_0^1 \rho \omega^2 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + \frac{C_2}{3} \int_0^1 \frac{\omega^4}{\rho} dx \leq C \int_0^1 \rho \theta^2 v^2 dx \end{aligned}$$

and after integration over $]0, t[, t > 0$, from (4.21) it follows

$$(4.22) \quad \begin{aligned} & \|\phi(., t)\|^2 + \|v(., t)\|_{L^4(\Omega)}^4 + \|\omega(., t)\|_{L^4(\Omega)}^4 \\ & \leq C \left(\int_0^t M_\rho(\tau) \max_{x \in \Omega} v^2 \|\theta(., \tau)\|^2 d\tau + \|\phi(., 0)\|^2 + \|v_0\|_{L^4(\Omega)}^4 + \|\omega_0\|_{L^4(\Omega)}^4 \right). \end{aligned}$$

Note that

$$(4.23) \quad v_0, \omega_0 \in L^4(\Omega), \quad \phi(., 0) \in L^2(\Omega).$$

Taking into account (4.6) and (3.34) from (4.22) we get

$$(4.24) \quad \|\phi(., t)\|^2 \leq C \left(\int_0^t \max_{x \in \Omega} v^2(x, t) \|\phi(., \tau)\|^2 d\tau + 1 \right).$$

With the help of (3.37) and Gronwall's inequality from (4.24) for each $t > 0$ we obtain

$$(4.25) \quad \|\phi(., t)\|^2 \leq C$$

and hence (4.4).

Because of (3.25) and (3.34) from (4.21) for $t > 0$ we obtain immediately

$$(4.26) \quad \int_0^t \left\| \frac{\partial \theta}{\partial x}(., \tau) \right\|^2 d\tau \leq C \left(\int_0^t \max_{x \in \Omega} v^2(x, \tau) \|\theta(., \tau)\|^2 d\tau + 1 \right).$$

Using (4.4) and (3.37) from (4.26) one can easily conclude that the inclusion (4.5) is true. \square

LEMMA 4.2. ([1], pp.93-94) *It holds*

$$(4.27) \quad \frac{\partial \rho}{\partial x} \in L^\infty(0, \infty; L^2(\Omega)),$$

$$(4.28) \quad \theta^{1/2} \frac{\partial \rho}{\partial x} \in L^2(0, \infty; L^2(\Omega)),$$

$$(4.29) \quad \frac{\partial v}{\partial x} \in L^2(0, \infty; L^2(\Omega)).$$

LEMMA 4.3. ([1], pp.94-95) *It holds*

$$(4.30) \quad \frac{\partial v}{\partial x} \in L^\infty(0, \infty; L^2(\Omega)), \quad \frac{\partial^2 v}{\partial x^2} \in L^2(0, \infty; L^2(\Omega)).$$

LEMMA 4.4. *It holds*

$$(4.31) \quad \frac{\partial \omega}{\partial x} \in L^\infty(0, \infty; L^2(\Omega)), \quad \frac{\partial^2 \omega}{\partial x^2} \in L^2(0, \infty; L^2(\Omega)),$$

$$(4.32) \quad \frac{\partial \theta}{\partial x} \in L^\infty(0, \infty; L^2(\Omega)), \quad \frac{\partial^2 \theta}{\partial x^2} \in L^2(0, \infty; L^2(\Omega)).$$

PROOF. We consider the equations (2.3) and (2.4) rewriting them in the next form:

$$(4.33) \quad \frac{1}{A} \frac{\partial \omega}{\partial t} = \frac{\partial \rho}{\partial x} \frac{\partial \omega}{\partial x} + \rho \frac{\partial^2 \omega}{\partial x^2} - \frac{\omega}{\rho},$$

$$(4.34) \quad \frac{\partial \theta}{\partial t} = -K \rho \theta \frac{\partial v}{\partial x} + \rho \left(\frac{\partial v}{\partial x} \right)^2 + \rho \left(\frac{\partial \omega}{\partial x} \right)^2 + \frac{\omega^2}{\rho} + D \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial x} + D \rho \frac{\partial^2 \theta}{\partial x^2}.$$

Multiplying (4.33) and (4.34) respectively by $\frac{\partial^2 \omega}{\partial x^2}$ and $\frac{\partial^2 \theta}{\partial x^2}$ and integrating over $]0, 1[$ we obtain

$$(4.35) \quad \frac{1}{2A} \frac{d}{dt} \int_0^1 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + \int_0^1 \rho \left(\frac{\partial^2 \omega}{\partial x^2} \right)^2 dx = \int_0^1 \frac{\omega}{\rho} \frac{\partial^2 \omega}{\partial x^2} dx - \int_0^1 \frac{\partial \rho}{\partial x} \frac{\partial \omega}{\partial x} \frac{\partial^2 \omega}{\partial x^2} dx,$$

$$(4.36) \quad \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{\partial \theta}{\partial x} \right)^2 dx + D \int_0^1 \rho \left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 dx$$

$$= K \int_0^1 \rho \theta \frac{\partial v}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx - \int_0^1 \rho \left(\frac{\partial v}{\partial x} \right)^2 \frac{\partial^2 \theta}{\partial x^2} dx - \int_0^1 \rho \left(\frac{\partial \omega}{\partial x} \right)^2 \frac{\partial^2 \theta}{\partial x^2} dx$$

$$- \int_0^1 \frac{\omega^2}{\rho} \frac{\partial^2 \theta}{\partial x^2} dx - D \int_0^1 \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx.$$

Taking into account the inequalities

$$(4.37) \quad \omega^2 \leq 2 \|\omega\| \left\| \frac{\partial \omega}{\partial x} \right\|, \quad \left(\frac{\partial \omega}{\partial x} \right)^2 \leq 2 \left\| \frac{\partial \omega}{\partial x} \right\| \left\| \frac{\partial^2 \omega}{\partial x^2} \right\| \leq 2 \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2$$

and applying on the right-hand side of (4.35) the Young inequality with a parameter $\varepsilon_i > 0$ ($i = 1, 2$) we get

$$(4.38) \quad \frac{1}{2A} \frac{d}{dt} \int_0^1 \left(\frac{\partial \omega}{\partial x} \right)^2 dx + \int_0^1 \rho \left(\frac{\partial^2 \omega}{\partial x^2} \right)^2 dx$$

$$\leq C \|\omega\| \left\| \frac{\partial^2 \omega}{\partial x^2} \right\| + \max_{x \in \Omega} \left| \frac{\partial \omega}{\partial x} \right| \left\| \frac{\partial \rho}{\partial x} \right\| \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|$$

$$\leq \varepsilon_1 \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C_{\varepsilon_1} \|\omega\|^2 + C \left\| \frac{\partial \omega}{\partial x} \right\|^{1/2} \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^{3/2} \left\| \frac{\partial \rho}{\partial x} \right\|$$

$$\leq \varepsilon_1 \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C_{\varepsilon_1} \|\omega\|^2 + \varepsilon_2 \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + C_{\varepsilon_2} \left\| \frac{\partial \rho}{\partial x} \right\|^4 \left\| \frac{\partial \omega}{\partial x} \right\|^2.$$

Let $\varepsilon_i > 0$ ($i = 1, 2$) be small enough. Integrating (4.38) over $]0, t[$ and using (2.13), (3.25), (3.37), (4.27) and (4.3) we conclude that

$$(4.39) \quad \sup_{t \in R^+} \left\| \frac{\partial \omega}{\partial x}(\cdot, t) \right\|^2 + \int_0^\infty \left\| \frac{\partial^2 \omega}{\partial x^2}(\cdot, \tau) \right\|^2 d\tau \leq C.$$

Therefore (4.31) holds true.

Taking into account (3.34), (4.37) and the inequalities

$$(4.40) \quad \left(\frac{\partial v}{\partial x}\right)^2 \leq 2\left\|\frac{\partial v}{\partial x}\right\|\left\|\frac{\partial^2 v}{\partial x^2}\right\| \leq 2\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2, \quad \left(\frac{\partial \theta}{\partial x}\right)^2 \leq 2\left\|\frac{\partial \theta}{\partial x}\right\|\left\|\frac{\partial^2 \theta}{\partial x^2}\right\|$$

and using the Young inequality with a parameter $\varepsilon_i > 0$ ($i = 1, 2, \dots, 5$), for the terms on right-hand side of (4.36) we find the estimates as follows:

$$(4.41) \quad \begin{aligned} \left|K \int_0^1 \rho \theta \frac{\partial v}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx\right| &\leq C \max_{x \in \Omega} \left|\frac{\partial v}{\partial x}\right| \|\theta\| \left\|\frac{\partial^2 \theta}{\partial x^2}\right\| \\ &\leq C \left\|\frac{\partial^2 v}{\partial x^2}\right\| \|\theta\| \left\|\frac{\partial^2 \theta}{\partial x^2}\right\| \\ &\leq \varepsilon_1 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_1} \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 \|\theta\|^2, \end{aligned}$$

$$(4.42) \quad \begin{aligned} \left|\int_0^1 \rho \left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial^2 \theta}{\partial x^2} dx\right| &\leq \varepsilon_2 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_2} \left\|\frac{\partial v}{\partial x}\right\|_{L^4(\Omega)}^4 \\ &\leq \varepsilon_2 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C \left\|\frac{\partial v}{\partial x}\right\|^2 \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2, \end{aligned}$$

$$(4.43) \quad \begin{aligned} \left|\int_0^1 \rho \left(\frac{\partial \omega}{\partial x}\right)^2 \frac{\partial^2 \theta}{\partial x^2} dx\right| &\leq \varepsilon_3 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_3} \left\|\frac{\partial \omega}{\partial x}\right\|_{L^4(\Omega)}^4 \\ &\leq \varepsilon_3 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_3} \left\|\frac{\partial \omega}{\partial x}\right\|^2 \left\|\frac{\partial^2 \omega}{\partial x^2}\right\|^2, \end{aligned}$$

$$(4.44) \quad \begin{aligned} \left|\int_0^1 \frac{\omega^2}{\rho} \frac{\partial^2 \theta}{\partial x^2} dx\right| &\leq \varepsilon_4 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_4} \|\omega\|_{L^4(\Omega)}^4 \\ &\leq \varepsilon_4 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_4} \|\omega\|^2 \left\|\frac{\partial \omega}{\partial x}\right\|^2, \end{aligned}$$

$$(4.45) \quad \begin{aligned} \left|\int_0^1 \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} dx\right| &\leq \max_{x \in \Omega} \left|\frac{\partial \theta}{\partial x}\right| \left\|\frac{\partial \rho}{\partial x}\right\| \left\|\frac{\partial^2 \theta}{\partial x^2}\right\| \\ &\leq C \left\|\frac{\partial \rho}{\partial x}\right\| \left\|\frac{\partial \theta}{\partial x}\right\|^{1/2} \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^{3/2} \\ &\leq \varepsilon_5 \left\|\frac{\partial^2 \theta}{\partial x^2}\right\|^2 + C_{\varepsilon_5} \left\|\frac{\partial \rho}{\partial x}\right\|^4 \left\|\frac{\partial \theta}{\partial x}\right\|^2. \end{aligned}$$

Let $\varepsilon_i > 0$ ($i = 1, 2, \dots, 5$) be small enough. Taking into account (4.41)–(4.45) and integrating (4.36) over $]0, t[$ we get

$$\begin{aligned}
 & \left\| \frac{\partial \theta}{\partial x}(\cdot, t) \right\|^2 + \int_0^t \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 d\tau \\
 & \leq C \left(\sup_{t \in R^+} \left\| \theta(\cdot, t) \right\|^2 \int_0^t \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 d\tau + \sup_{t \in R^+} \left\| \frac{\partial v}{\partial x}(\cdot, t) \right\|^2 \int_0^t \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 d\tau \right. \\
 & \quad + \sup_{t \in R^+} \left\| \frac{\partial \omega}{\partial x}(\cdot, t) \right\|^2 \int_0^t \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 d\tau + \sup_{t \in R^+} \left\| \omega(\cdot, t) \right\|^2 \int_0^t \left\| \frac{\partial \omega}{\partial x} \right\|^2 d\tau \\
 & \quad \left. + \sup_{t \in R^+} \left\| \frac{\partial \rho}{\partial x}(\cdot, t) \right\|^4 \int_0^t \left\| \frac{\partial \theta}{\partial x} \right\|^2 d\tau \right) + \left\| \frac{d}{dx} \theta_0 \right\|^2.
 \end{aligned} \tag{4.46}$$

Using the inclusions (2.13), (4.3), (4.4), (4.2), (4.5), (4.27), (4.30) and (4.31) from (4.46) we obtain (4.32). \square

LEMMA 4.5.

$$\frac{\partial \rho}{\partial t} \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; L^2(\Omega)), \tag{4.47}$$

$$\frac{\partial v}{\partial t}, \frac{\partial \omega}{\partial t}, \frac{\partial \theta}{\partial t} \in L^2(0, \infty; L^2(\Omega)). \tag{4.48}$$

PROOF. Taking into account (3.34), from the equation (2.1) we get immediately

$$\left\| \frac{\partial \rho}{\partial t}(\cdot, t) \right\|^2 \leq C \left\| \frac{\partial v}{\partial x}(\cdot, t) \right\|^2 \tag{4.49}$$

and using (4.30) and (4.29) we obtain (4.47).

With the help of the property ([1], pp.95)

$$\theta^2(x, t) \leq C [\theta(x, t) + \left\| \frac{\partial \theta}{\partial x}(\cdot, t) \right\|^2], \tag{4.50}$$

the estimate (3.34) and inequality (4.40), from (2.2) we get

$$\begin{aligned}
 \left\| \frac{\partial v}{\partial t} \right\|^2 & \leq C \left(\left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 \left\| \frac{\partial \rho}{\partial x} \right\|^2 + \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 \right. \\
 & \quad \left. + \int_0^1 \left(\frac{\partial \rho}{\partial x} \right)^2 \theta dx + \left\| \frac{\partial \theta}{\partial x} \right\|^2 \left\| \frac{\partial \rho}{\partial x} \right\|^2 + \left\| \frac{\partial \theta}{\partial x} \right\|^2 \right).
 \end{aligned} \tag{4.51}$$

Integrating (4.51) over $]0, t[$ and using (4.30), (4.27), (4.28) and (4.5) we conclude that

$$\int_0^t \left\| \frac{\partial v}{\partial t}(\cdot, \tau) \right\|^2 d\tau \leq C \tag{4.52}$$

for each $t > 0$.

With the help of (4.37), (3.34) and (3.25) from (2.3) we find that

$$(4.53) \quad \left\| \frac{\partial \omega}{\partial t} \right\|^2 \leq C \left[\left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 \left\| \frac{\partial \rho}{\partial x} \right\|^2 + \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + \|\omega\|^2 \right]$$

and taking into account (4.27), (4.31) and (4.2), for each $t > 0$ we have

$$(4.54) \quad \int_0^t \left\| \frac{\partial \omega}{\partial t}(\cdot, \tau) \right\| d\tau \leq C.$$

Finally, using (3.34), (4.40) and (4.37), from the equation (2.4) we find that

$$(4.55) \quad \begin{aligned} \left\| \frac{\partial \theta}{\partial t} \right\|^2 \leq & C \left(\left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 \|\theta\|^2 + \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 \left\| \frac{\partial v}{\partial x} \right\|^2 + \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 \left\| \frac{\partial \omega}{\partial x} \right\|^2 \right. \\ & \left. + \left\| \frac{\partial \omega}{\partial x} \right\|^2 \|\omega\|^2 + \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 \left\| \frac{\partial \rho}{\partial x} \right\|^2 + \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2 \right). \end{aligned}$$

Using the inclusions (4.30), (4.31), (4.32), (4.2), (4.3), (4.4) and (4.27), from (4.55) we get easily

$$(4.56) \quad \int_0^t \left\| \frac{\partial \theta}{\partial t}(\cdot, \tau) \right\|^2 d\tau \leq C.$$

□

LEMMA 4.6. *It holds*

$$(4.57) \quad \frac{\partial \rho}{\partial x} \in L^2(0, \infty; L^2(\Omega)).$$

PROOF. From the equations (2.1) and (2.2) it follows

$$(4.58) \quad \frac{\partial}{\partial t} \left(\frac{\partial \ln \rho}{\partial x} \right) + K \theta \frac{\partial \rho}{\partial x} = - \frac{\partial v}{\partial t} - K \rho \frac{\partial \theta}{\partial x}.$$

Multiplying (4.58) by $\frac{\partial \ln \rho}{\partial x}$ and integrating over $]0, 1[$ we obtain

$$(4.59) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \ln \rho}{\partial x} \right\|^2 + K \int_0^1 \rho \theta \left(\frac{\partial \ln \rho}{\partial x} \right)^2 dx \\ = - \int_0^1 \frac{\partial v}{\partial t} \frac{\partial \ln \rho}{\partial x} dx - K \int_0^1 \rho \frac{\partial \theta}{\partial x} \frac{\partial \ln \rho}{\partial x} dx. \end{aligned}$$

The second term at the right-hand side we estimate with the help of the Cauchy inequality:

$$(4.60) \quad \begin{aligned} \rho \left| \frac{\partial \theta}{\partial x} \frac{\partial \ln \rho}{\partial x} \right| & \leq \rho \left[\frac{1}{2} \theta \left(\frac{\partial \ln \rho}{\partial x} \right)^2 + \frac{1}{2\theta} \left(\frac{\partial \theta}{\partial x} \right)^2 \right] \\ & \leq \rho \left[\frac{\theta}{2} \left(\frac{\partial \ln \rho}{\partial x} \right)^2 + \frac{1}{4} \left(1 + \frac{1}{\theta^2} \right) \left(\frac{\partial \theta}{\partial x} \right)^2 \right]. \end{aligned}$$

Taking into account (4.60) and integrating (4.59) over $]0, t[$ we get

$$(4.61) \quad \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, t) \right\|^2 + K \int_0^t \int_0^1 \rho \theta \left(\frac{\partial \ln \rho}{\partial x} \right)^2 dx d\tau \leq 2 \int_0^t \left\| \frac{\partial v}{\partial t} \right\| \left\| \frac{\partial \ln \rho}{\partial x} \right\| d\tau \\ + \frac{K}{2} \int_0^t \int_0^1 \rho \left(\frac{\partial \theta}{\partial x} \right)^2 dx d\tau + \frac{K}{2} \int_0^t \int_0^1 \frac{\rho}{\theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 dx d\tau + \left\| \frac{d \ln \rho_0}{dx} \right\|^2.$$

Applying the Young inequality with a parameter $\varepsilon > 0$ on the first integral on the right-hand side and using the properties (4.5), (3.8), (3.34), (3.25) and (2.13) from (4.61) we find that

$$(4.62) \quad \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, t) \right\|^2 + C \int_0^t m_\theta(\tau) \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, \tau) \right\|^2 d\tau \\ \leq \varepsilon \int_0^t \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, \tau) \right\|^2 d\tau + C_\varepsilon \int_0^t \left\| \frac{\partial v}{\partial t}(\cdot, \tau) \right\|^2 d\tau + C.$$

With the help of (3.46) we obtain

$$(4.63) \quad C \int_0^t \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, \tau) \right\|^2 d\tau \leq \sup_{t \in R^+} \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, t) \right\|^2 \int_0^t f_1(\tau) d\tau \\ + \varepsilon \int_0^t \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, \tau) \right\|^2 d\tau + C_\varepsilon \int_0^t \left\| \frac{\partial v}{\partial t}(\cdot, \tau) \right\|^2 d\tau + C,$$

where f_1 is defined by (3.48). Taking ε small enough and using (4.27) and (4.48) from (4.63), for each $t > 0$ we find that

$$(4.64) \quad \int_0^t \left\| \frac{\partial \ln \rho}{\partial x}(\cdot, \tau) \right\|^2 d\tau \leq C.$$

Using (3.25) from (4.64) we get immediately (4.57). \square

Theorem 2.1 is an immediate consequence of the above lemmas.

REMARK. In the second paper we intend to prove (by using Theorem 2.1) that a solution of the problem (2.1)–(2.11) converges to a stationary solution $(\alpha^{-1}, 0, 0, E_1)$ (where E_1 and α are defined by (3.1) and (3.9) respectively) in the space $(H^1(\Omega))^4$, when $t \rightarrow \infty$.

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